Engineering Maths First Aid Kit

8.5

The chain rule

Introduction

The **chain rule** is used when it is necessary to differentiate a function of a function.

This rule is summarised here.

1. The chain rule

Consider the function $y = (\sin x)^3$. This process involves cubing the function $\sin x$.

Consider also the function $y = \log_e(x^3 + 5x)$. Here we are finding the logarithm of the function $x^3 + 5x$.

In both cases we are finding a **function of a function.**

The chain rule is used to differentiate such composite functions and is illustrated in the examples which follow.

Example

Find $\frac{dy}{dx}$ when $y = \sin(5x + 3)$.

Solution

Notice that $5x + 3$ is a function of x, so $\sin(5x + 3)$ is a function of a function.

To simplify the problem we can introduce a new variable z and write $z = 5x + 3$ so that y becomes

$$y = \sin z$$

Then, differentiating this with respect to z,

$$\frac{dy}{dz} = \cos z$$

Now, in fact, we want $\frac{dy}{dx}$. The chain rule states

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

So

$$\frac{dy}{dx} = \cos z \times 5 \qquad \text{since} \qquad \frac{dz}{dx} = 5$$

Then, finally

$$\frac{dy}{dx} = 5\cos z = 5\cos(5x + 3)$$

The chain rule: if $y(z)$ is a function of z and $z(x)$ is a function of x, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \times \frac{\mathrm{d}z}{\mathrm{d}x}$$

Example

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \mathrm{e}^{(x^2)}$.

Solution

x^2 is a function, so $\mathrm{e}^{(x^2)}$ is a function of a function. If we let $z = x^2$, then $y = \mathrm{e}^z$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 2x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = \mathrm{e}^z$$

so that, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \times \frac{\mathrm{d}z}{\mathrm{d}x} = \mathrm{e}^z \times 2x = 2x\mathrm{e}^{(x^2)}$$

Example

If $y = \sin^3 x$ find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

First of all note that $\sin^3 x$ means $(\sin x)^3$. Therefore y can be written $y = (\sin x)^3$, so that this is a function of a function.

If we let $z = \sin x$ then $y = z^3$. It follows that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \cos x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = 3z^2$$

Then, using the chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}z} \times \frac{\mathrm{d}z}{\mathrm{d}x} = 3z^2 \times \cos x = 3\sin^2 x \cos x$$

Exercises

In each case find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

1. $y = \sin(x^2)$.
2. $y = (\sin x)^2$.
3. $y = \log_{\mathrm{e}}(x^2 + 1)$.
4. $y = (2x + 7)^8$
5. $y = \mathrm{e}^{2x-3}$

Answers

1. $2x\ \cos(x^2)$. 2. $2\sin x \cos x$. 3. $\frac{2x}{x^2+1}$. 4. $16(2x + 7)^7$. 5. $2\mathrm{e}^{2x-3}$.

